

A Model of Vanishing Cosmological Constant

Aiichi Iwazaki

*International Politics and Economics, Nishogakusha University,
6-16 3-bantyo Chiyoda Tokyo 102-8336, Japan.*

(Dated: Jun. 25, 2012)

We propose a model in which there exists a real scalar field q satisfying a condition $\dot{q} = MH$ and its energy density is given by $\frac{1}{2}\dot{q}^2 + V(q)$, where H is the Hubble parameter ($H = \dot{a}/a$) and M is a mass scale characterizing the field. We show that the potential $V(q)$ of the field is uniquely determined by the condition. The potential depends on the energy densities of background matters. We find that the vacuum energy of the matters is cancelled by the potential of the field. As a result, the minimum of the total energy density of the matters and the field vanishes and is located at the infinite scale factor $a = \infty$. This remarkable property results without a supersymmetry. We show that the present tiny dark energy is caused by early inflation, while the energy is comparable to Planck scale before the inflation. As our model is reduced to the Λ CDM model in the limit $M \rightarrow 0$, it is a natural generalization of the Λ CDM model.

PACS numbers: 95.36.+x, 98.80.Cq,
Cosmological Constant, Dark Energy, Quintessence, Inflation

I. INTRODUCTION

Recent observations[1] have strongly suggested the acceleration of the cosmic expansion at present. The acceleration would be caused by the cosmological constant or a fluid with the equation of state $\omega \simeq -1$. It is an intriguing idea that the cosmological constant is not constant but varying with time. Hence, much attentions[2] have recently been paid to such fluids describing the dynamical cosmological constant. There is, however, no guiding principle for the construction of a promising model of the dynamical cosmological constant. For instance, in models using scalar fields describing the cosmological constant, various potentials of the fields have been proposed, but they are plagued with many free parameters.

Models of dynamical cosmological constant must satisfy observational constraints[3], but the constraints are not sufficiently stringent so as to uniquely determine a promising model. Furthermore, there are serious problems to be solved. One of the problems is why the vacuum energy density $\sim (10^{-3}\text{eV})^4$ at present is extremely smaller than the natural scale $\sim (\text{Planck mass})^4$. The other one is why the cosmic expansion begins to accelerate nearly at the present epoch. At least, the expansion of the Universe does not accelerate around the epoch of the recombination. It is a coincidence that we observe the acceleration of the cosmic expansion at the present epoch. The solutions to these problems could not be found so far in previous models, although the coincident problem is alleviated in models with tracking fields[4]. (So called "tracking condition" for the potentials of the scalar fields can be a guiding principle for the construction of models. But there are a variety of potentials[5, 6] satisfying the condition. Thus, it is not restrictive so that a promising model is uniquely determined. Furthermore, the vacuum energy is forced to vanish by tuning parameters in these models. Thus, the cosmological constant problem is not solved.)

In this paper we propose a guiding principle for the construction of a specific model of dynamical cosmological constant. The principle is that there exists a real scalar field q satisfying a condition $\dot{q} = MH$ where H is the Hubble parameter ($H = \dot{a}/a$) and M is a mass scale characterizing the field. We also assume that its energy density is given by $\frac{1}{2}\dot{q}^2 + V(q)$. Then, we can uniquely determine the potential $V(q)$ of the field. It is consisted of two parts, an intrinsic part and a part depending on the energy densities of background matters. The intrinsic potential is an ordinary potential depending only on the field, while the other potential depends on the energy densities of the matters. This latter property gives rise to a remarkable property that the minimum of the total energy density (the energy densities of q and the matters) vanishes. Namely, the vacuum energy vanishes as if there is a supersymmetry.

In the next section II we explain our assumptions as well as our guiding principle for the construction of dynamical cosmological constant. We derive the potential of the field q by using the principle. In the section III we show that the total energy density of matters and the field vanishes at $a = \infty$. Namely, the vacuum energy vanishes. In section IV we analyze the dynamical properties of the field q . In section V we discuss observational constraints on parameters in our model. As a result we find that only one parameter M remains as a free parameter. We also find that the model is reduced to the Λ CDM in the limit of $M \rightarrow 0$. In section VI we discuss that the small dark energy at present is caused by inflation. In section VII we discuss how the cosmological constant and coincidence problems are alleviated in our model. In section VIII we discuss our principle and summarize our results.

II. OUR MODEL

We consider a spatial homogeneous flat space Universe with the scale factor $a(t)$; $ds^2 = dt^2 - a(t)^2 d\vec{x}^2$. We suppose that a real scalar field q exists as well as matters (radiations or non-relativistic matter) in the Universe. Now we explain our assumptions. The first of all, the field obeys the field equation,

$$\ddot{q} + 3H\dot{q} + \partial_q V = 0 \quad (1)$$

with the Hubble parameter $H = \dot{a}/a$, where V denotes the potential of the field. (The dot denotes a derivative in t). The potential is determined by a condition we propose in this paper. The condition is given in the following,

$$\frac{\dot{q}}{M} = \frac{\dot{a}}{a} = H, \quad (2)$$

where M denotes a mass scale characterizing the field. Namely, the scalar field characterizes the expansion rate of the Universe. Furthermore, we assume an Einstein equation with the gravitational constant G such that

$$H^2 = \frac{8\pi G(\rho_m + \rho_q)}{3} = \frac{\rho_m + \rho_q}{m^2} \quad (3)$$

with $m^2 \equiv 3/8\pi G$, as well as the continuity equation

$$\dot{\rho}_m = -3H(\rho_m + p_m), \quad (4)$$

where ρ_m (p_m) denotes the energy density (pressure) of the matters. Here we assume that the energy density ρ_q of the field q is given by $\rho_q \equiv \dot{q}^2/2 + V$. If we impose the continuity equation on the field q , that is, $\dot{\rho}_q = -3H(\rho_q + p_q)$ with $p_q \equiv \dot{q}^2/2 - V$, it gives rise to the field equation (1). Thus it is not independent of the equation (1).

Without the condition eq(2), the field is a component in the Universe and do not play any special roles. But as we show below, with the condition the potential V of the field must depend on the energy density of the matters. In the sense the field is a very special component in the Universe.

Now, we derive the potential V of the scalar field q by using the condition in eq(2). First, we rewrite the Einstein equation in(3) such that

$$H^2 = \frac{\rho_q + \rho_m}{m^2} = \frac{\dot{q}^2/2 + V + \rho_m}{m^2} = \frac{M^2 H^2/2 + V + \rho_m}{m^2}. \quad (5)$$

Thus, we can express the Hubble parameter such that

$$H^2 = \frac{2(V + \rho_m)}{2m^2 - M^2}, \quad (6)$$

where it is natural to take $2m^2 - M^2 > 0$; the mass M characterizing the field q is less than the Planck scale $\sim m$.

Furthermore, using the field equation $\ddot{q} + 3H\dot{q} + \partial_q V = M\dot{H} + 3MH^2 + \partial_q V = 0$, we obtain

$$\dot{q}(M\dot{H} + 3MH^2 + \partial_q V) = \frac{d}{dt}\left(\frac{M^2 H^2}{2} + V\right) + 3M^2 H^3 = 0, \quad (7)$$

where we used the condition $\dot{q} = MH$.

Inserting the Hubble parameter in eq(6) into this equation, we obtain

$$2a \frac{d}{da} V + 6\lambda V = -\lambda \left(a \frac{d}{da} \rho_m + 6\rho_m \right), \quad (8)$$

with $\lambda \equiv M^2/m^2$, where we used the relation $d/dt = \dot{a} d/da$.

This is the equation to determine the potential V when the energy density ρ_m of the matters is given as a function of a . For example in the non-relativistic matter (we simply call it as matter) dominated period when ρ_m is given such as $\rho_m \equiv \rho_0 a^{-3}$, we find

$$V = \frac{\lambda \rho_m}{2(1-\lambda)} + V_0 a^{-3\lambda} \quad \text{in the matter dominated period} \quad (9)$$

where V_0 is a constant determined by the present values of the density parameters Ω_q and Ω_m ; $\Omega_q = \rho_q/(\rho_q + \rho_m)$ and $\Omega_m = \rho_m/(\rho_q + \rho_m)$. Similarly, we obtain the potential in the radiation dominated period; $V = \lambda \rho_m/(4-3\lambda) + V_0 a^{-3\lambda}$ where $\rho_m \propto a^{-4}$. In general the solution is given by

$$V = \frac{\lambda}{2} \left(-\rho_m + a^{-3\lambda} \int da (3\lambda - 6) a^{3\lambda-1} \rho_m(a) \right) + V_0 a^{-3\lambda}. \quad (10)$$

In this way, using our condition in eq(2), we can determine the potential of the field q . We should stress that the potential is consisted of two terms; one is the term depending on the energy density of the background matters and the other is the intrinsic term $V_{\text{int}} \equiv V_0 a^{-3\lambda}$.

III. ZERO VACUUM ENERGY

In general, we have zero point vacuum energies in quantum field theories. They are infinite so that we need proper renormalization. For example in the standard model of quarks and leptons, we appropriately renormalize the vacuum energy. Because its absolute value is not observable in the Euclidean space without gravity, we may take its value arbitrary. (The relative value of the vacuum energy is observable, e.g. Casimir effect.) But, the vacuum energy is observable when we take account of gravity. We cannot take it arbitrary. Thus, we may renormalize the vacuum energy such that it is just given by the cosmological constant observed at present. The vacuum energy density ($\sim (10^{-3}\text{eV})^4$) determined in this way is fairly small compared with the other scales in the standard model with gravity. If there is a supersymmetry, the vacuum energy exactly vanishes. but the symmetry is broken at the scale Λ beyond 1 TeV. Then, the energy density would be of the order of Λ^4 , which is much larger than the observed one $\sim (10^{-3}\text{eV})^4$. We do not know why such large discrepancy is present. The difficulty of the question comes partly from the fact that we have no mechanism controlling the vacuum energy.

Our condition postulated in this paper can give a mechanism for rendering the vacuum energy to vanish. Indeed, we can see that the solution $V(a)$ vanishes as $a \rightarrow \infty$ if $\rho_m \rightarrow 0$ as $a \rightarrow \infty$. Therefore, the Hubble parameter H or $V + \rho_m$ vanishes in the limit $a \rightarrow \infty$. It implies that the total energy density of the matter and the field vanishes in the limit. Because $H^2 \geq 0$ and $\dot{H} < 0$ (as we show later), the minimum of the total energy density is located at $a = \infty$. Even if we add a constant c to ρ_m , the solution $V(a)$ in eq(9) or eq(10) has a term $-c$ so that the total energy density proportional to $V + \rho_m$ does not change. Thus, the minimum remains to vanish. This is a general result caused by the condition in eq(2). Therefore, the condition leads to the vanishing vacuum energy.

For clarity we give an example of a classical real scalar field as a background matter. We consider a massless scalar field whose energy density is given by $\rho_m \equiv \dot{\phi}^2/2 + v$, where v is a constant. The equation of motion of the field, $\ddot{\phi} + 3H\dot{\phi} = 0$, can be easily solved. The solution is given such that $\dot{\phi} = ca^{-3}$ where c is a constant. Thus, the energy density ρ_m is given by $c^2 a^{-6}/2 + v$. Because the energy density is obtained as a function of a , the potential V of the field q can be found from eq(10) such that $V = -v + V_0 a^{-3\lambda}$. Therefore, we have the total energy density $\simeq V + \rho_m = c^2 a^{-6}/2 + V_0 a^{-3\lambda}$, which has no constant term proportional to v and vanishes at $a = \infty$. We find that the total vacuum energy of the matter field ϕ and the field q vanishes, even if the vacuum energy of the field ϕ takes any value,

Previous models of dynamical cosmological constant have been plagued by the ambiguity of the vacuum energy. That is, as we cannot properly treat the zero point energies in field theories without supersymmetry, there is the ambiguity of the precise value of the vacuum energies in ρ_m . But there is no such ambiguity in our model. As long as our principle holds, the total energy density should vanish in the vacuum even if we do not know the correct vacuum energy of the matter fields. This is a prominent property in our model.

IV. q FIELD

The potential V in eq(9) was obtained as a function of the scale factor a , not the field q . But it is easy to rewrite V in terms of q . Because we explicitly obtain $q = M \log(a/a_0)$ from the condition eq(2), the potential V can be rewritten such that

$$\bar{V}(q) \equiv V(a = a_0 \exp(q/M)) = \frac{\lambda \rho_0 a_0^{-3} \exp(-\frac{3q}{M})}{2(1-\lambda)} + V_0 a_0^{-3\lambda} \exp(-\lambda \frac{3q}{M}) \quad \text{in the matter dominated period,} \quad (11)$$

where a_0 is an integration constant and represents the period when the field q vanishes.

The potential $\propto \exp(-3\lambda q/M)$ intrinsic for q should be of the order of M^4 . This is because when q takes the value of M , the potential should be the order of M^4 . (As we see later, our model is reduced to the Λ CDM model in the limit of $\lambda \rightarrow 0$. The parameter λ is found much smaller than unity by observations.) Thus, we may assume $V_0 a_0^{-3\lambda} = M^4$. Then, it follows that

$$\bar{V}(q) = \frac{\lambda \bar{\rho}_0 \exp(-\frac{3q}{M})}{2(1-\lambda)} + M^4 \exp(-\lambda \frac{3q}{M}) \quad \text{in matter dominated period,} \quad (12)$$

with $\bar{\rho}_0 \equiv \rho_0 a_0^{-3}$.

In this way we can derive the explicit form of the potential $\bar{V}(q)$ from the condition in eq(2). The potential is composed of two terms; one is the term proportional to $\rho_m = \bar{\rho}_0 \exp(-\frac{3q}{M})$ (the first term in eq(12)) and the other one $V_{\text{int}} \equiv M^4 \exp(-3\lambda q/M)$ is the term intrinsic for the field q (the second term in eq(12)). The former depending on ρ_m is much smaller than ρ_m itself because $\lambda \ll 1$ (see later). We should stress that the intrinsic term $V_{\text{int}}(a) \propto a^{-3\lambda}$ of the potential decreases more slowly than the first term $\propto \rho_m(a) \propto a^{-3}$ as a increases. Therefore, the total energy density $\rho_q(a) + \rho_m(a)$ eventually approaches $V_{\text{int}}(a)$ with the expansion of the Universe. That is, the field q dominates the Universe. Thus, we may regard the present value $V_{\text{int}}(a=1) = V_0$ as the dark energy observed.

We note that the equation of the state of the field q is given by

$$\omega_q = \frac{p_q}{\rho_q} = \frac{\frac{\dot{q}^2}{2} - V}{\frac{\dot{q}^2}{2} + V} = \frac{\lambda \rho_m + 2(\lambda - 1)V}{\lambda \rho_m + (\lambda + 2)V} \simeq \frac{\lambda \rho_m - 2V}{\lambda \rho_m + 2V} \quad (13)$$

with $\lambda \ll 1$, where we used eq(2) and eq(6). Therefore, we find that when the field q dominates the Universe, the expansion accelerates because $\lambda \rho_m$ is smaller than V so that $\omega_q \simeq -1$. We mention that with small $\lambda \ll 1$, the potential in eq(12) satisfies "tracking condition"[7] $\bar{V}''(q)\bar{V}(q)/\bar{V}'(q)^2 > 1$, where $\bar{V}' \equiv \partial_q \bar{V}$ and $\bar{V}'' \equiv \partial_q^2 \bar{V}$.

The potential in eq(12) has been discussed in previous papers [6], in which more general ones than the potential $\bar{V}(q)$ have been considered such that $A \exp(-\alpha q/M) + B \exp(-\beta q/M)$. The parameters A, B, α, β and M have been partially determined so as to obtain a quintessence model with favorable "tracking fields". But free parameters still remain in these potentials[8], although the parameters are partially fixed by observations. Other models[5] with various potentials have been proposed, but they are also plagued with many free parameters. On the other hand, there is only one free parameter in our model. Actually, we have three parameters M, ρ_0 and V_0 . Among them, ρ_0 and V_0 are determined by the present values of the density parameters Ω_q and Ω_m . The remaining one M is only one adjustable parameter, which can be determined by observing the equation of state of the field q today. Although our model involves only one adjustable parameter M or λ , the model is shown to be consistent with current observations as long as we have small λ ($\ll 1$).

We should mention a peculiar property of the field q . In general, when we solve a field equation like eq(1), we need to impose initial conditions. They may be arbitrarily taken. But this is not true for the field q . An initial condition for \dot{q}_{in} at $t = t_{\text{in}}$ is determined by the condition $\dot{q}_{\text{in}} = M H_{\text{in}}$ in eq(2) where H_{in} is Hubble parameter at $t = t_{\text{in}}$. On the other hand, an initial condition for q_{in} at $t = t_{\text{in}}$ can be arbitrarily taken. The arbitrary choice of a value q_{in} corresponds to determining the value of a_0 ; $q_{\text{in}} = M \log(a_{\text{in}}/a_0)$. In this way, the dynamics of the field q is fairly restrictive compared with ordinary scalar fields in the other models of quintessence. This restriction comes from the condition in eq(2). It claims that the field q is not independent of the scale factor a , although the field equation of q is required to hold.

We can see the actual behavior of the field q by examining the equation of motion,

$$\ddot{q} = -\partial_q \bar{V}(q) - 3H\dot{q} = \frac{3\lambda \bar{\rho}_0 \exp(-\frac{3q}{M})}{2M(1-\lambda)} + \frac{3\lambda M^4 \exp(-\frac{3\lambda q}{M})}{M} - \frac{3\dot{q}^2}{M}, \quad (14)$$

where the last term in the right hand side acts as a friction. Therefore, a particle with the coordinate q slowly goes down the slope of the potential, being affected by the friction; the coordinate q logarithmically increases with time because $q = M \exp(a/a_0)$ and $a \propto t^{2/3}$ in the matter dominated period. The initial condition $\dot{q}_{\text{in}} = MH_{\text{in}} > 0$ implies that the particle never goes up the slope; it must always goes down. This implies that the Universe always expands as long as it expands at the initial period $t = t_{\text{in}}$. Although the result holds in the matter dominated period, the similar result is obtained in the radiation dominated period.

It is important to see that $\ddot{q} < 0$, because the right hand side of the equation can be rewritten in the following,

$$\ddot{q} = \frac{3}{M} \left(\frac{\lambda \rho_m}{2(1-\lambda)} + \lambda V_{\text{int}} - \frac{2\lambda(V + \rho_m)}{2-\lambda} \right) = \frac{3}{M} \left(-\frac{\lambda \rho_m}{2(1-\lambda)} - \frac{\lambda^2 V_{\text{int}}}{2-\lambda} \right) < 0, \quad (15)$$

with $\lambda \ll 1$, where we used $\dot{q}^2/2 = \lambda(V + \rho_m)/(2-\lambda)$ and $V = \lambda \rho_m/2(1-\lambda) + V_{\text{int}}$ in the matter dominated period. The same inequality holds in the radiation dominated period. Hence, the Hubble parameter $H = \dot{q}/M$ always decreases with time; $\dot{H} < 0$. In other words the total energy density always decreases with time. Thus, its minimum ($= 0$) is located at $a = \infty$. That is, the vacuum energy vanishes in our model.

It is also instructive to see that the coordinate q linearly increases with time in the period of inflation. This is because the scale factor a exponentially increases in the period. In this way, the solution q of the equation represents the expansion history of the Universe.

V. OBSERVATIONAL CONSTRAINTS

We wish to determine the parameters in the potential V by examining observational constraints. First we consider the density parameters Ω_q , Ω_m and the equation of state ω_q in the matter dominated period. They are given by respectively,

$$\Omega_q = \frac{2V + \lambda \rho_m}{2(V + \rho_m)} = \frac{2(1-\lambda)V_0 a^{-3\lambda} + (2-\lambda)\lambda \rho_m}{2(1-\lambda)V_0 a^{-3\lambda} + (2-\lambda)\rho_m} \simeq \frac{V_0 a^{-3\lambda} + \lambda \rho_0 a^{-3}}{V_0 a^{-3\lambda} + \rho_0 a^{-3}}, \quad (16)$$

$$\Omega_m = \frac{(2-\lambda)\rho_m}{2(V + \rho_m)} = \frac{(2-\lambda)(1-\lambda)\rho_m}{2(1-\lambda)V_0 a^{-3\lambda} + (2-\lambda)\rho_m} \simeq \frac{\rho_0 a^{-3}}{V_0 a^{-3\lambda} + \rho_0 a^{-3}}, \quad (17)$$

$$\omega_q = \frac{p_q}{\rho_q} = \frac{\frac{\dot{q}^2}{2} - V}{\frac{\dot{q}^2}{2} + V} = \frac{-2(1-\lambda)^2 V_0 a^{-3\lambda}}{2(1-\lambda)V_0 a^{-3\lambda} + \lambda(2-\lambda)\rho_m} \simeq \frac{-V_0 a^{-3\lambda}}{V_0 a^{-3\lambda} + \lambda \rho_0 a^{-3}}, \quad (18)$$

where we used $\dot{q}^2/2 = \lambda(V + \rho_m)/(2-\lambda)$ and took the limit $\lambda \ll 1$.

Using the present values $\Omega_q^0 = 0.75$ and $\Omega_m^0 = 0.25$ at $a = 1$ (here we take $a = 1$ as the scale factor today), we obtain the parameters $V_0 = 0.75\rho_c^0$ and $\rho_0 = 0.25\rho_c^0$ where ρ_c^0 denotes the critical density $\rho_c^0 \equiv \rho_0 + V_0$ at present. Furthermore, we find that

$$\omega_q(a \simeq 1) \simeq -1 + \frac{4\lambda}{3} + \lambda(1-a), \quad (19)$$

where we have taken the limit $a \rightarrow 1$. Current observations suggests $\omega_q(a = 1) \lesssim -0.9$. It leads to the constraint $\lambda \lesssim 0.08$. As our formula has only one parameter λ , we can check the validity[3] of our model by observing $\omega_q(a \simeq 1)$ in detail.

We can examine whether or not the field q affects Big Bang Nucleosynthesis, which occurs at about $a = 10^{-9}$. (As radiations dominates the Universe at the period, ρ_m in the above formulas eq(16) is formally replaced by $\rho_m/2$ where ρ_m is the energy density of radiations proportional to a^{-4} .) Because $\Omega_q \simeq \lambda$ for sufficiently small $a \ll 1$ such as $\lambda a^{-4} \gg a^{-3\lambda}$, the field does not affect the Nucleosynthesis[3, 9] if $\lambda < 0.1$. Therefore, our model satisfies both of the observational constraints with the small parameter $\lambda < 0.1$.

We should mention that when we take the limit $M \rightarrow 0$ ($\lambda \rightarrow 0$), the field $\dot{q} = MH$ vanishes and the total energy density of the Universe becomes $\rho_0 a^{-3} + V_0$ in the matter dominated period; $\dot{q}^2/2 + V(a) \rightarrow V_0$ in the limit $M \rightarrow 0$. Thus, our model is reduced to the standard Λ CDM model with the cosmological constant $\Lambda = 3V_0/m^2$. (Here we take the limit keeping the critical density today $\rho_c^0 = \rho_0 + V_0$ fixed.) Because the standard Λ CDM model is consistent with current observations, our model is also consistent with them as long as the parameter $\lambda = M^2/m^2$ is much small.

It is interesting to see that the energy density of the field dominates the Universe only around the present epoch $a = 1$. In order to see it we calculate

$$\frac{\Omega_q}{\Omega_m} = \frac{2(1-\lambda)V_0a^{-3\lambda} + (2-\lambda)\lambda\rho_m}{(2-\lambda)(1-\lambda)\rho_m} \simeq \frac{V_0a^{-3\lambda} + \lambda\rho_m}{\rho_m} = \frac{\frac{V_0}{\rho_0}a^{-3\lambda} + \lambda a^{-3}}{a^{-3}} = \lambda + \frac{V_0}{\rho_0}a^{3(1-\lambda)} \simeq \lambda + 3a^3 \quad (20)$$

with $\lambda \ll 1$, where we used the present values $\Omega_q^0 = 3\Omega_m^0$ or $V_0 = 3\rho_0$. Thus, we find that the field q dominates the Universe just after the period $a = (1/3)^{1/3} \simeq 0.7$.

More precisely we estimate the turning point from the deceleration to the acceleration of the Universe, that is, the period when $\ddot{a} = 0$. The period is obtained by solving the equation,

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{\rho_m + \rho_q + 3p_q}{2m^2} = \frac{2(V + \rho_m) + 3\lambda(V + \rho_m) - 3V(2-\lambda)}{2m^2(2-\lambda)} \\ &= \frac{(-4 + 6\lambda)V + (2 + 3\lambda)\rho_m}{2m^2(2-\lambda)} \simeq \frac{(\rho_0(2 + 3\lambda) + V_0(-4 + 6\lambda)a^{3-3\lambda} - 2\rho_0\lambda)a^{-3}}{2m^2(2-\lambda)} = 0. \end{aligned} \quad (21)$$

Thus, it follows that

$$\frac{\ddot{a}}{a} \propto \frac{\rho_0}{V_0}(2 + \lambda) + (-4 + 6\lambda)a^{3-3\lambda} = \frac{\rho_0}{V_0}(2 + \lambda) + (-4 + 6\lambda)\frac{(1+z)^{3\lambda}}{(1+z)^3} = 0, \quad (22)$$

with $a = 1/(1+z)$. As $\rho_0/V_0 = 1/3$ derived from the present density parameters, we find that the turning point from the deceleration to the acceleration is given by

$$\ddot{a}(z) = 0 \quad \text{at} \quad z \simeq 0.8 \quad (0.82) \quad \text{for} \quad \lambda = 0.1 \quad (\lambda = 0.01), \quad (23)$$

while $\ddot{a} = 0$ at $z(a) \simeq 0.82$ in the Λ CDM model obtained by taking the limit $\lambda = M^2/m^2 \rightarrow 0$.

We have shown that using the condition in eq(2) we uniquely determine the model of the dynamical cosmological constant. Our model involves only one free parameter λ and is consistent with the current observations when $\lambda < 0.1$. We have also shown that the model is reduced to the Λ CDM in the limit $\lambda \rightarrow 0$. Therefore, our model is a natural extension of the standard Λ CDM.

VI. INFLATION

We now proceed to show why the present value of dark energy $V_{\text{in}}(a = 1) = V_0 = M^4 a_0^{3\lambda}$ is much smaller than the natural scale m^4 of the Universe. In other words, we show why a_0 is extremely small such as $a_0 = (V_0/M^4)^{1/3\lambda} \sim 10^{-400}$ with $V_0 = (10^{-3}\text{eV})^4$ and $\lambda \sim 0.1$. The period at $a = a_0$ is the epoch when the field q vanishes; $q = M \log(a/a_0)$. Without inflation the scale factor a at most smoothly increases such as t^n with positive n being $O(1)$. It never reaches the value $a = 1$ within the life time of the Universe. Thus, we suppose that the period $a = a_0$ is earlier than the beginning of inflation[10]. Then, by the inflation the scale factor a exponentially increases and can reaches $a = 1$. The inflation makes the dark energy $V_{\text{in}} \propto a^{-3\lambda}$ exponentially small, while the potential V_{in} takes the large value $M^4 \simeq \lambda^2 m^4$ at $a = a_0$ before the inflation. Thus, the present small dark energy is realized, that is, $V_{\text{in}}(a) = V_0 a^{-3\lambda} = M^4 (a_0/a)^{3\lambda} \rightarrow V_0$ as $a \rightarrow 1$.

In order to show how it can be actually realized in our model, we may explicitly take $\rho_{\text{inf}} = ka^\epsilon$ as a flat potential of inflaton[10] with $0 < |\epsilon| \ll 1$. Then, ρ_{inf} is almost constant in a (ka_i^ϵ is a constant of the order of M'^4 , in which the inflation is supposed to begins at $a = a_i$ and M' is a typical scale of inflaton; $M' < M$.) The precise form of the inflaton potential is not necessary for the later discussion. Only what we need is the flatness of the potential in a . So, the parameter ϵ must be extremely small. We also assume that $k\epsilon$ is positive in order for the total energy density to be positive, see later. (We need extremely small but non vanishing parameter ϵ , because if $V_{\text{inf}} \propto a^\epsilon$ is exactly constant in a , i.e. $\epsilon = 0$, the constant V_{inf} is cancelled by the term $-V_{\text{inf}}$ in V so that it does not contribute to total energy density.)

We assume that the kinetic energy of the inflaton is much small compared with the potential energy when it rolls along the slope of the flat potential. Similarly, the kinetic energy of the field q is small compared with its potential

as we have discussed in previous sections. Therefore, we may assume that the total energy density is given by the potentials of the field q and the inflaton, The potential $V(a)$ is obtained by solving eq(8) with ρ_m replaced by ρ_{inf} ,

$$V = -\frac{k(1 + \frac{\epsilon}{6})}{1 + \frac{\epsilon}{3\lambda}}a^\epsilon + V_{\text{in}} \simeq -ka^\epsilon + \frac{k\epsilon a^\epsilon}{3\lambda} + V_0 a^{-3\lambda} \quad (24)$$

where we used that $\epsilon \ll \lambda$. Thus, the Hubble parameter is given by

$$H^2 = \frac{2(V + \rho_{\text{inf}})}{2m^2 - M^2} \simeq (1 + \frac{\lambda}{2})m^{-2} \left(\frac{k\epsilon a^\epsilon}{3\lambda} + V_0 a^{-3\lambda} \right). \quad (25)$$

where the first term in the right hand side of the equation represents the contribution of the inflaton and the second term does the contribution of the field q . As we pointed out, the factor $k\epsilon$ must be positive. As the second term decreases with the expansion of the Universe much faster than the first term, the first term eventually dominates the Hubble parameter H . The inflation begins at $a = a_i \gg a_0$ where a_i satisfies

$$\frac{k\epsilon a_i^\epsilon}{3\lambda} = \frac{M'^4 \epsilon}{3\lambda} \simeq V_0 a_i^{-3\lambda} = M^4 \left(\frac{a_0}{a_i} \right)^{3\lambda}, \quad (26)$$

That is, $a_i \simeq a_0 \left(\frac{M'^4 \lambda}{M^4 \epsilon} \right)^{1/3\lambda}$. Once the inflation starts, the potential $V_{\text{in}}(a) = M^4(a_0/a)^{3\lambda}$ rapidly decreases with time. Therefore, we have extremely small potential $V_{\text{in}}(a = a_f) = M^4(a_0/a_f)^{3\lambda}$ at the end $a = a_f (\gg a_i)$ of the inflation compared with the potential $V_{\text{in}}(a = a_i) = M^4(a_0/a_i)^{3\lambda}$ at the beginning $a = a_i$ of the inflation. This is because the ratio $V_{\text{in}}(a_i)/V_{\text{in}}(a_f) = (a_f/a_i)^{3\lambda}$ is exponentially large. In other words the dark energy can be much small at the end of the inflation, although it is not necessarily small at the beginning of the inflation. In this way we can obtain much small dark energy at present.

We may estimate e-folding number of the inflation leading to the small dark energy today. As the present value V_0 is of the order of $(10^{-3}\text{eV})^4$, the e-folding number N of the inflation may be roughly estimated such that

$$N = \log \frac{a_f}{a_i} \sim 920, \quad \text{because} \quad \frac{V_{\text{in}}(a_i)}{V_{\text{in}}(a_f = 1)} = \left(\frac{a_i}{a_f = 1} \right)^{-3\lambda} = \left(\frac{a_i}{a_f = 1} \right)^{-0.3} \sim \left(\frac{M}{10^{-3}\text{eV}} \right)^4 \sim 10^{120}, \quad (27)$$

with $\lambda = 0.1$, where we assumed that the inflation begins at $a_i = a_0$ and ends at $a_f = 1$. Obviously, this e-folding number is much larger than the actual one because the inflation begins after the period a_0 ($< a_i$) and ends much earlier than today $a_f \ll 1$. The above estimation is an example to show that the inflation really makes the potential much small by taking an appropriate value of the e-folding number N . In this way we can have much small intrinsic potential energy at the end of the inflation. (The reason why a_0 is much small compared with the present value $a = 1$ is obvious. Such a small a is exponentially inflated. Then, it smoothly increases to reach the present value $a = 1$ with the standard expansion of the Universe.)

We would like to point out the behavior of q in the period of the inflation. As we obtained the potential V in eq(24), we can easily solve the equation (14) for q . Inserting $V \simeq -ka^\epsilon = -ka_0^\epsilon \exp(\epsilon q/M)$ into the equation, $\ddot{q} + 3H\dot{q} + \partial_q V(q) \simeq 3\dot{q}^2/M - k\epsilon a_0^\epsilon/M = 0$, we obtain a solution such that $\dot{q} = \sqrt{k\epsilon a_0^\epsilon/3}$. Thus, we find that the field q linearly increases with time. This implies that the scale factor a exponentially increases with time.

Finally, we point out that it is difficult to realize the above inflation scenario by means of real scalar fields. To see it, we consider a massive real scalar field described by the Lagrangian $L = \frac{1}{2}(\dot{\phi}^2 - \mu^2\phi^2)$. The equation of motion of the field $\ddot{\phi} + 3H\dot{\phi} + \mu^2\phi = 0$ can be solved by assuming the slow roll motion of the field, i.e. $\ddot{\phi} \ll \mu^2\phi$ in the period of the inflation when H is a constant given by h . The solution is given by $\phi \propto \exp(-\mu^2 t/3h) = (\exp ht)^{(-\mu^2/3h^2)}$. Thus, we have $\phi = ca^{-\mu^2/3h^2}$ with a constant c because $a \propto \exp(ht)$ in the period of the inflation. The energy density of the field is given such that $\rho_m = ka^{-2\mu^2/3h^2}$ with $k \equiv c^2\mu^2/2$ in term of the scale factor a . The slow roll condition $\ddot{\phi} \ll \mu^2\phi$ is satisfied when the inequality $\mu^2 \ll h^2$ holds. Namely, we obtain the flat inflaton potential ka^ϵ with $\epsilon \equiv -2\mu^2/3h^2$; $|\epsilon| \ll 1$. It seems apparently that the flat potential leads to the exponential increase of the scale factor $a \sim \exp(ht)$, that is the inflation. But this is not true in our case where the square of the Hubble parameter is proportional to $V + \rho_m$, not to ρ_m . We find from eq(25) that the consistency condition $k\epsilon > 0$ must hold. However, the our model of the inflaton does not satisfy this condition. Indeed we have $k\epsilon < 0$; $k = c^2\mu^2/2 > 0$ and $\epsilon = -2\mu^2/3h^2 < 0$. In this way, the simple inflation model using the real scalar field does not work. The difficulty of making naive inflaton model comes from the absence of the constant term in the total energy density.

VII. COSMOLOGICAL CONSTANT AND COINCIDENCE PROBLEMS

We would like to discuss both problems of the cosmological constant and coincidence in our model.

First we discuss the problem of the cosmological constant. The problem is why the vacuum energy density (V_{in} in our model) at present is much small compared with a natural scale $m^4 \sim (\text{Planck mass})^4$ given by gravity. If we had approximate supersymmetry in the standard model, such a small energy density can be explained when the breaking scale of the supersymmetry is such small as 10^{-3}eV . But the real breaking scale is beyond 1 TeV.

On the other hand, as if there is a supersymmetry in our model, the vacuum energy vanishes owing to the condition eq.(2). Namely, even if there are any quantum corrections of matter fields for vacuum energy, the corrections are cancelled by the field q so that the total vacuum energy vanishes. Owing to this remarkable property, the small vacuum energy density at present can be realized by the inflation. That is, it tremendously dilutes the large vacuum energy density of the order of $M^4 = m^4\lambda^2$. Although we still need a tuning of parameters to obtain the appropriate small energy density $(10^{-3}\text{eV})^4$ at present, the problem of the cosmological constant is fairly alleviated.

The coincidence problem is why the vacuum energy density is comparable to the energy density of the dark matter at present. In our model this corresponds to the question why $V_0 = V_{\text{in}}(a=1) \sim \rho_m(a=1) = \rho_0$. In order to discuss the problem we note the equation(20),

$$\frac{\Omega_q}{\Omega_m} \simeq \lambda + (V_0/\rho_0)a^3 \quad \text{in the non relativistic matter dominated period} \quad (28)$$

where we took the limit $\lambda \ll 1$. This implies that for the arbitrarily chosen value V_0/ρ_0 , the equality $\Omega_q \sim \Omega_m$ arises at the epoch $a = (\rho_0/V_0)^{1/3}$. After the epoch, the field q dominates the Universe. As explained above, the inflation tremendously dilutes the vacuum energy density V_{in} . After the inflation, $V_{\text{in}} (\propto a^{-3\lambda})$ much slowly decreases with the expansion of the Universe because $\lambda \ll 1$. On the other hand, the energy density of the matter $\rho_m \propto a^{-3}$ or a^{-4} decreases much faster than V_{in} . Therefore, the coincidence problem can be described as how the inflation dilutes the vacuum energy density such that the ratio V_0/ρ_0 is of the order of unity.

In this way both solutions for the cosmological constant and coincidence problems lie in the detail of inflation model: How does the inflation brings out extremely small vacuum energy $V_0 \simeq (10^{-3}\text{eV})^4$ and renders the ratio V_0/ρ_0 of the order of unity?

VIII. SUMMARY AND DISCUSSION

To summarize, postulating the presence of the real scalar field q satisfying the condition $\dot{q}/M = H$, we have derived the simple analytical model of the dynamical cosmological constant. The model has several remarkable properties. The most remarkable one is the property of the zero vacuum energy. Because the potential of the field q depends on the energy densities of background matters, even if any quantum corrections of the matters are added to the vacuum energy, they are cancelled by the potential of the field q . In addition, the total energy density decreases with the expansion of the Universe and vanishes at $a = \infty$. Therefore, the vacuum energy vanishes even if any ordinary matters are present in the Universe.

Furthermore, we have shown that inflation tremendously decreases the dark energy density V_{in} because $V_{\text{in}} \propto a^{-3\lambda}$. Thus, the inflation can make the energy density extremely small, while it is comparable to the scale m^4 or $M^4 = \lambda^2 m^2$ before the inflation. As a result the dark energy density becomes much small such as $(10^{-3}\text{eV})^4$ at present. The coincidence problem is addressed by the question how the inflation gives rise to the ratio of the order of unity between the dark energy $V_{\text{in}}(a=1)$ and the dark matter $\rho_m(a=1)$, that is, $V_{\text{in}}(a=1)/\rho_m(a=1) = V_0/\rho_0 \sim O(1)$. Hence, to answer the problem, we need to examine the detail of the inflation mechanism. As there is only one adjustable parameter λ involved in our model, we can check the model by observing the equation of state $\omega_q(a \sim 1)$ at present. Our model is reduced to the Λ CDM model in the limit $\lambda \rightarrow 0$. Thus, our model is a natural generalization of the Λ CDM model.

We have proposed the relation $\dot{q} = MH$ between the field q and the Hubble parameter H . But we do not know how the relation can be incorporated in a Lagrangian formalism. Furthermore, we do not know generalized equations describing the spatially inhomogeneity of q . Probably, there would be a symmetry behind our model, which guarantees the vanishing vacuum energy. If the Lagrangian formalism is obtained, we will be able to find the symmetry as well as the equations describing the spatially inhomogeneity of the field q . In our model, we assume the relation in addition to the ordinary equations, i.e. Einstein equations with matters and the field equation of q . Thus, It seems apparently that our model is consistent with the Einstein gravity. But the relation could not be incorporated in the Einstein

gravity. Therefore, our model would be a key for pursuing more fundamental theory of gravity, which has the property of the vanishing vacuum energy without supersymmetry.

-
- [1] A. G. Riess et al., *Astron. J.* **116**, 1009 (1998).
 S. Perlmutter et al., *Astrophys. J.* **517**, 565 (1999).
 E. Komatsu et al., *Astrophys. J. Suppl.* **180**, 330 (2009).
 W. J. Percival et al., *Mon. Not. Roy. Astron. Soc.* **401**, 2148 (2010).
 - [2] see reviews, S. Tsujikawa, astro-ph/10041493.
 K. Bamba, S. Capozziello, S. Nojiri and S.D. Odintsov, astro-ph/1205.3421.
 A. Silvestri and M. Trodden, *Rept. Prog. Phys.* **72**, 096901,(2009).
 R. Rakhi, K. Indulekha, astro-ph/09105406.
 R.R. Caldwell, R. Dave and P.J. Steinhardt, *Phys. Rev. Lett.* **80** 1582 (1998).
 C. Armendariz-Picon, Viatcheslav F. Mukhanov, and Paul J. Steinhardt, *Phys. Rev. Lett.* **85** 4438 (2000).
 R. R. Caldwell, *Phys. Lett.* **B545** 23 (2002).
 - [3] M. Kowalski et al., *Astrophys. J.* **686**, 749 (2008).
 Daniel J. Eisenstein et al., *Astrophys. J.* **633**, 560 (2005).
 J. Dunkley et al., *Astrophys. J. Suppl.* **180**, 306 (2009).
 I. Maor, R. Brustein, J. McMahon and P. J. Steinhardt, *Phys. Rev. D* **65**, 123003 (2002).
 - [4] I. Zlatev, L. Wang and P. J. Steinhardt, *Phys. Rev. Lett.* **82**, 896,(1999).
 - [5] B. Ratra, and P.J.E. Peebles, *Phys. Rev. D* **37**, 3406 (1988).
 C. Wetterich, *Nuclear Physics B* **302**, 668 (1988).
 P.G. Ferreira, and M. Joyce, *Phys. Rev. Lett.* **79**, 4740 (1997).
 P.G. Ferreira and M. Joyce, *Phys. Rev. D* **58**, 023503 (1998).
 P. Brax and J. Martin *Phys. Rev. D* **61**, 103502 (2000).
 A. Albrecht and C. Skordis *Phys. Rev. Lett.* **84**, 2076 (2000).
 - [6] T. Barreiro, E. J. Copeland and N. J. Nunes, *Phys. Rev. D* **61**, 127301, (2000).
 C. Rubano, P. Scudellaro, E. Piedipalumbo, S. Capozziello and M. Capone, *Phys. Rev. D* **69**, 103510 (2004).
 - [7] P. J. Steinhardt, L. M. Wang and I. Zlatev, *Phys. Rev. D* **59**, 123504 (1999).
 - [8] A. Albrecht and C. Skordis, *Phys. Rev. Lett.* **84**, 2076 (2000).
 V. Sahni and L. Wang, *Phys. Rev. D* **62** (2000) 103517.
 S. Dodelson, M. Kaplinghat, E. Stewart, *Phys. Rev. Lett.* **85** 5276 (2000).
 - [9] R. Bean, S. H. Hansen and A. Melchiorri, *Phys. Rev. D* **64**, 103508 (2001).
 R.H. Cyburt, B. Fields, K. Olive and E. Skillman, *Astropart. Phys.* **23**, 313 (2005).
 - [10] A. R. Liddle and D. H. Lyth, "Cosmological inflation and large-scale structure", Cambridge University Press (2000).